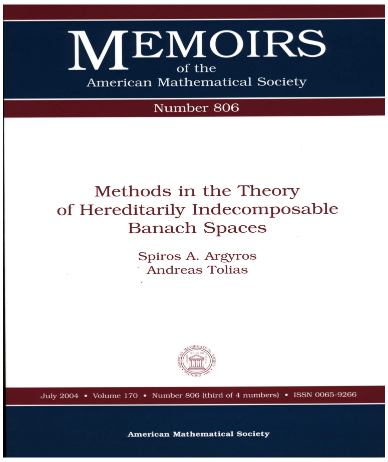
“ΠΡΟΒΛΗΜΑΤΑ ΔΙΑΣΠΑΣΗΣ ΚΑΙ ΒΕΛΤΙΣΤΗΣ ΠΡΟΣΕΓΓΙΣΗΣ ΣΕ ΧΩΡΟΥΣ BANACH”

Σ.Α. ΑΡΓΥΡΟΣ, Β. ΚΑΝΕΛΛΟΠΟΥΛΟΣ, Α. ΑΡΒΑΝΙΤΑΚΗΣ, Α. ΤΟΛΙΑΣ

ΣΧΟΛΗ Ε.Μ.Φ.Ε ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ



FEATURED REVIEW. *

This memoir deals with structural properties of general Banach spaces; in the ’90s, several questions going back to S. Banach’s book [*Théorie des opérations linéaires*, Sub. Fund. Narodowej, Warsaw, 1932; Zbl 0005.20901] were solved, and new objects like hereditarily indecomposable (H.I.) spaces appeared; their behavior under duality and quotients as well as their relations with the usual concepts of reflexivity and separability have been investigated since then, and a lot of results have been obtained; many of them can be found in this memoir.

An infinite sequence (e_n) in a Banach space is a basic sequence if it is a Schauder basis of its closed linear span; every (infinite dimensional) Banach space contains a basic sequence (proved in Banach’s book). Two basic sequences (e_n) and (f_n) are equivalent if the map T defined by $T(e_n) = f_n$ extends to a Banach isomorphism of their closed linear spans; a basic sequence (e_n) is unconditional if it is equivalent to $(\pm e_n)$, for all possible independent choices of signs. The unconditional basic sequence problem asked whether every Banach space contains such a sequence. Another question considered in the ’60s was whether every Banach space contains a basic sequence equivalent to the unit vector basis of l_p for some $p \in [1, \infty)$ or of c_0. This was disproved by B. S. Tsirel′son [Funktsional. Anal. i Prilozhen. **8** (1974), no. 2, 57–60; MR0350378 (50 #2871)], who constructed a space T with basis, whose dual unit ball is described by a clever inductive process, so that

$$\|\sum_{k=1}^{n} b_k\|_T \geq \frac{1}{2}\sum_{k=1}^{n} \|b_k\|_T$$

for every n and any blocks $(b_k)_{k=1}^{n}$ in T located after the place n, and furthermore, so that the constant $1/2$ is optimal in a very strong sense; this forbids l_1-subspaces by well-known results of R. C. James, and clearly T cannot contain c_0 or l_p for $p > 1$. Tsirel′son’s solution remained the basis for nearly all the further developments in that direction.

However, the (negative) solution of the unconditional basic sequence problem did not use the space T directly, but a variation S introduced by T. Schlumprecht [Israel J. Math. **76** (1991), no. 1-2, 81–95; MR1177333 (93h:46023)] that has a basis invariant under spreading that “optimally” satisfies

$$\|\sum_{k=1}^{n} b_k\| \geq \sum_{k=1}^{n} \|b_k\| / \ln_2(n+1)$$

for every n and any n blocks, giving more flexibility than the constant factor $1/2$. Breaking unconditionality actually led to H.I. spaces [W. T. Gowers and B. Maurey, J. Amer. Math. Soc. **6** (1993), no. 4, 851–874; MR1201238 (94k:46021)]: a Banach space X is H.I. if the unit spheres of any two infinite dimensional subspaces Y and Z of X almost intersect,

$$\inf\{\|y - z\| : y \in Y, z \in Z, \|y\| = \|z\| = 1\} = 0,$$

and this obviously prevents unconditional sequences from existing in X. The fact that breaking unconditionality leads to H.I. spaces was better understood after Gowers’ dichotomy theorem: every Banach space contains an unconditional basic sequence or an H.I. subspace [W. T. Gowers, Geom. Funct. Anal. **6** (1996), no. 6, 1083–1093; MR1421876 (97m:46017)]. On an H.I. space there are few operators: every operator on a complex H.I. space has the form $\lambda \mathrm{Id} + S$, with S strictly singular; by Fredholm theory, it follows that an H.I. space is not isomorphic to any proper subspace; this solves a question about linear dimension in Banach’s book, a variant of which was known as the hyperplane problem.

This smallness of $\mathcal{L}(X)$ was used to get a few unusual examples in theories related to Banach space theory. Let us mention some points that are not touched upon by the memoir. The K-theory of Banach algebras helped verify the properties of more examples of spaces having, in some sense, small spaces of operators; conversely, H.I.-type spaces were used to find algebras of the form $\mathcal{L}(X)$ with peculiar K_0 and K_1 groups [N. J. Laustsen, J. London Math. Soc. (2) **59** (1999), no. 2, 715–728; MR1709676 (2000f:46093); A. Zsák, Proc. London Math. Soc. (3) **84** (2002), no. 3, 747–768; MR1888430 (2003c:46084)]. Let us also mention connections with Fredholm-like theories [P. Aiena and M. González, Math. Z. 233 (2000), no. 3, 471–479; MR1750932 (2001f:47015)].

In a series of papers, Argyros (with various coauthors) has replaced the use of the Schlumprecht space by a technique closer to that of Tsirel′son: a sequence $E_1, \dots, E_n$ of n finite subsets of $\mathbb{N}$ is admissible if $n \leq \min E_1$ and $\max E_k < \min E_{k+1}$ for $k = 1, \dots, n-1$; then, if (e_j) denotes the basis in T, one has, for $E = \bigcup_{k=1}^{n} E_k$ and all scalars (c_j),

$$\|\sum_{j \in E} c_j e_j\|_T \geq \frac{1}{2}\sum_{k=1}^{n} \|\sum_{j \in E_k} c_j e_j\|_T.$$

The starting point of the generalized Tsirel′son constructions is to define notions of admissibility of increasing ordinal complexity (Schreier families). This variable complexity gives a flexibility replacing that of S. This technique yields several new facts that cannot be obtained via the original S-technique, in particular for the treatment of non-separable spaces. The first one is an H.I. space that is also asymptotically l_1 [S. A. Argyros and I. Deliyanni, Trans. Amer. Math. Soc. **349** (1997), no. 3, 973–995; MR1390965 (97f:46021)]; there are also the striking interpolation results of Argyros and V. Felouzis [J. Amer. Math. Soc. **13** (2000), no. 2, 243–294 (electronic); MR1750954 (2002b:46021)], among many other results.

The memoir under review contains a thorough study of the relations between the H.I. property and duality, quotients, non-separability, and reflexivity. The authors start with the definition of a ground G and a norming set D, and describe a general method for constructing a space X_G based on G and D; varying the properties of the ground and norming set, they recover essentially all the results obtained earlier by the generalized T-technique. One main example summarizes the situation: there exists an H.I. space Y with Y^* separable and H.I., and Y^{**} non-separable and H.I. The bounded operators on Y^{**} are of the form $\lambda\,\mathrm{Id} + R$, where R has separable range. This bidual Y^{**} exhibits properties similar to famous examples of Shelah and Shelah-Steprans.

The memoir is organized as follows: after a detailed introduction, a first chapter of a general nature about H.I. spaces; Chapter 2: Schreier families; Chapter 3: definition of a ground G, norming set D and the general space associated to them; Chapter 4: proof of the basic inequality; Chapters 5, 6: technical tools for H.I. constructions, especially rapidly increasing sequences; Chapter 7 uses the machinery to build an H.I. space X_G; Chapter 8: the predual $(X_G)_*$ is also H.I.; Chapter 9: operators on X_G are $\lambda \mathrm{Id} + S$, with S strictly singular and weakly compact, when the ground G is $\mathcal{S}_2$-bounded; Chapter 10: defining G to make the dual X_G^* non-separable and H.I.; Chapter 11: complemented embedding of l_p in duals of H.I.; Chapter 12: compact families in $\mathbb{N}$; Chapter 13: case of an S_ξ-bounded ground G; Chapter 14: quotients of H.I. spaces: in particular, every separable space not containing l_1 is the quotient of an H.I. space.

Several additional results have appeared since the writing of the memoir: Argyros, J. López-Abad and S. B. Todorčević [C. R. Math. Acad. Sci. Paris **337** (2003), no. 1, 43–48; MR1993994 (2004f:46018)] construct a reflexive non-separable Banach space isomorphic to no proper subspace and with “few” operators; Argyros and A. Tolias [Geom. Funct. Anal. **14** (2004), no. 2, 247–282; MR2060194 (2005d:46021)] give a space X that is unconditionally saturated but has an H.I. dual X^*.

Reviewed by *Bernard Maurey*

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S.A. ARGYROS AND A. TOLIAS
INDECOMPOSABILITY AND UNCONDITIONALITY IN DUALITY
GEOMETRIC AND FUNCTIONAL ANALYSIS VOL. 14 (2004) 247-282

PAPER’S REVIEW

This deep article establishes the existence of a remarkable infinite-dimensional Banach space X_{uh}, which enjoys the following properties: X_{uh} is reflexive, every infinite-dimensional closed subspace contains an unconditional basic sequence, and the dual space X_{uh}^* is hereditarily indecomposable (HI). It follows in particular that X_{uh} is indecomposable (in other words, is not the direct sum of two closed infinite-dimensional subspaces), although it is saturated with unconditional sequences. Such an example had been constructed in a previous work of the first author and A. Manoussakis [Studia Math. **159** (2003), no. 1, 1–32;; MR2030739 (2005d:46022)], and the present work relies in part on refinements of the techniques used in this Argyros-Manoussakis article. Another straightforward consequence of the main result is that the operator spaces $B(X_{uh})$ and $B(X_{uh}^*)$ are isomorphic, although one of the spaces is unconditionally saturated and the other is HI. It is also shown that every operator T on X_{uh}^* can be written as $T = \lambda I + S$ where S is a strictly singular operator.

Duality shows that every quotient of X_{uh} is indecomposable. But it is also shown in this work that every quotient of X_{uh} has a further quotient which is HI. This last result provides the first example of a reflexive space X which is unconditionally saturated and has HI quotients, while a theorem due to E. Odell asserts that this cannot happen when X has an unconditional finite-dimensional decomposition.

Reviewed by *Gilles Godefroy*

ΛΟΙΠΕΣ ΔΗΜΟΣΙΕΥΣΕΙΣ

3. S.A.ARGYROS, A.ARVANITAKIS REGULAR AVERAGING AND REGULAR EXTENSION OPERATORS IN WEAKLY COMPACT SUBSETS OF HILBERT SPACES SERDICA MATH..J. 30 (2004) N0.4 527-548

4. A.ARVANITAKIS A PROOF OF THE GENERALIZED BANACH CONTRACTION CONJECTURE PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY 131 (2003) N0.12 3647-3656

5. V.KANELLOPOULOS A PROOF OF W.T. GOWERS’ c_0 THEOREM PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY VOL.132 NO..11 3231-3242

6. V.KANELLOPOULOS RAMSEY FAMILIES OF SUBTREES OF THE DYADIC TREE TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY (TO APPEAR)

7. Α. ΤΟΛΙΑΣ ΜΕΛΕΤΗ ΤΗΣ ΔΟΜΗΣ ΚΑΘΟΛΙΚΑ ΑΔΙΑΣΠΑΣΤΩΝ ΧΩΡΩΝ BANACH ΚΑΙ ΤΟΥ ΧΩΡΟΥ ΤΩΝ ΤΕΛΕΣΤΩΝ ΤΟΥΣ ΔΙΔΑΚΤΟΡΙΚΗ ΔΙΑΤΡΙΒΗ ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ