

Stochastic Analysis and Reliability Assessment of Shells

Research Team

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1. Introduction

The analysis of shells presents a challenge, since their formulation may become cumbersome and their behavior can be unpredictable with regard to geometry or support conditions. For this reason, shells have been considered as the “prima donna of structures” in the sense that their performance depends very much on how they are designed and how they are treated. The extreme sensitivity of thin shells to imperfections in material, geometry and boundary conditions can be described reasonably well, if the assumption of a random fluctuation of these imperfections is induced in the analysis, in other words if a stochastic finite element analysis is performed.

2. Finite element analysis of shells with multiple random material and geometric properties

In the first part of the present project, a stochastic finite element analysis of shell structures with combined uncertain material and geometric properties is performed. For this purpose, a stochastic formulation of the triangular composite facet shell element TRIC is derived assuming random variation of the Young’s modulus, the Poisson’s ratio and the thickness of the shell. As a result of the proposed formulation and the special features of the element, the stochastic stiffness matrix of TRIC depends finally on a minimum number of random variables representing the stochastic field (local average and weighted integral methods). This fact leads to a cost-effective stiffness matrix, which is very important in the case of a time consuming stochastic analysis of real world structures (Stefanou et al. [1]).

The spatial fluctuation of the mechanical and geometric properties is described by uncorrelated 2D-1V homogeneous Gaussian stochastic fields, sample functions of which are generated via the spectral representation method (Figure 1). Under the assumption of a pre-specified power spectral density function for these stochastic fields, it is possible to compute the response variability of a realistic shell structure using the direct Monte Carlo Simulation (MCS) technique. For the simulation of the stochastic fields, the Karhunen-Loeve expansion could also be used combined with a wavelet-

Galerkin scheme for the efficient numerical solution of the respective Fredholm integral equation (eigenvalue problem). However, some numerical instabilities reported during the calculation of eigenvalues at various wavelet levels (Stefanou et al. [4]) led finally to the use of the spectral representation method.

In this section, two shells of cylindrical (Scordelis-Lo shell) and hyperboloid shape are tested (Stefanou et al. [1]). The sensitivity of response statistics with regard to the scale of correlation of the stochastic fields, quantified via the correlation length parameter b is first examined. For both shell geometries, the displacement variability shows similar trends, starting from small values for small correlation lengths, up to large values for large correlation lengths. When the Young's modulus is the only random parameter of the problem, the coefficient of variation (COV) of the selected displacement w_c at a characteristic point C of each structure tends to the COV of Young's modulus ($\sigma_E=10\%$) for large values of parameter b , as expected. It is also evident that random variation in the shell thickness has significant effect on the displacement variability compared to the effect of random Young's modulus. When both quantities are supposed to vary randomly, the response COV tends to values that are 2-2.5 times greater than the input COV (Figure 2).

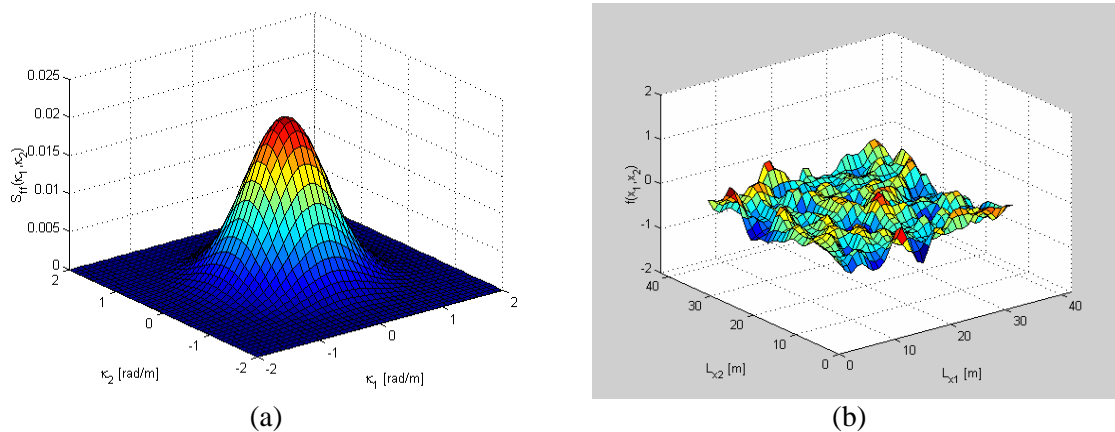


Figure 1. (a) Power spectral density function (b) Sample function pattern of a stochastic field produced by the spectral representation method.

In the case of combined fluctuation of Young's modulus and Poisson's ratio, the displacement variability curve indicates the small effect of the Poisson's ratio on the results. Slight differences are found between the results given by the local average method and those of the weighted integral method in the case of Scordelis-Lo shell. These differences become somewhat more pronounced in the hyperboloid shell case (Figure 3). In addition, the effect of type of correlation, expressed in this work by the mathematical form of the power spectrum, is investigated. Due to the relation existing between the power spectral density function of the stochastic field describing the random input quantities and the variance of a response quantity, slight differences are once again observed.

Investigations similar to the aforementioned were also made with regard to the stress variability. The main observation in this case is that the stress fluctuation is substantially different not only between several stress components but even between the

same stress component within different elements. This variation takes its maximum value when the Young's modulus and the shell thickness are simultaneously varying.

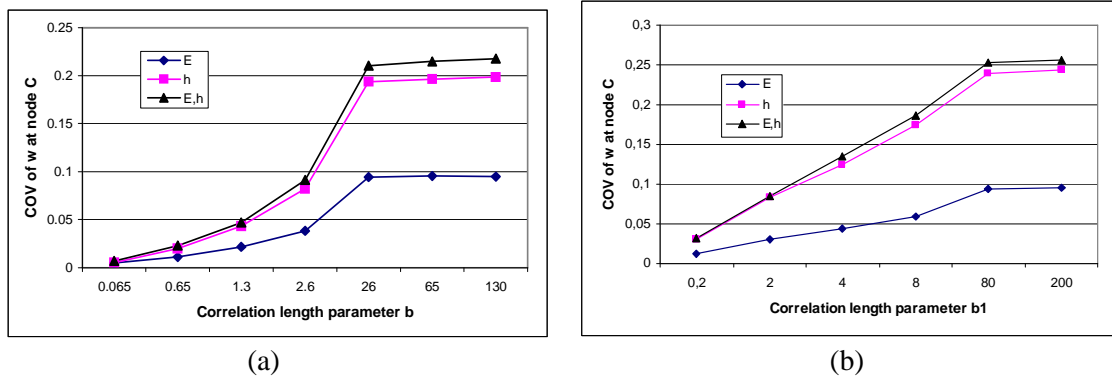


Figure 2. (a) Scordelis-Lo shell: COV of displacement w_c as a function of correlation length parameter b (b) Hyperboloid shell: COV of displacement w_c as a function of correlation length parameter b_1 ($\sigma_E = \sigma_h = 10\%$).

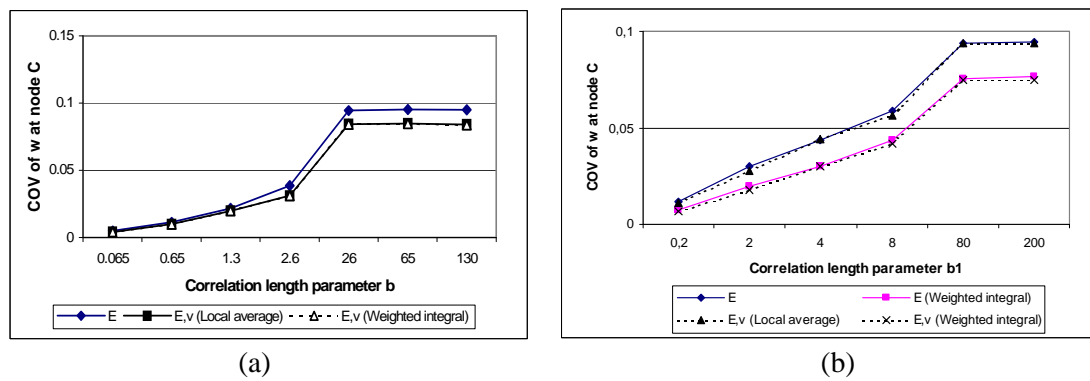


Figure 3. (a) Scordelis-Lo shell (b) Hyperboloid shell: Comparison between local average and weighted integral methods – COV of displacement w_c as a function of correlation length parameter for the case of combined Young's modulus and Poisson's ratio variation ($\sigma_E = \sigma_\nu = 10\%$).

3. Improving the computational efficiency in stochastic shell finite element analysis

The Monte Carlo Simulation (MCS) technique, which is the most effective and widely applicable method for handling large-scale stochastic Finite Element (FE) problems with complicated structural response, involves expensive computations due to the successive FE analyses required. Consequently, the need for developing efficient computational techniques emerges, in order to accelerate the MCS procedure and make it more tractable in structural engineering practice. The second part of the work performed in the framework of the present project is focused on the acceleration of the most time consuming tasks performed in MCS-based stochastic FE analysis of shell structures (Charpis et al. [2,3]), as described below.

3.1 Generation of stochastic field samples

Stochastic FE analysis can be performed using two separate meshes: (a) *a stochastic mesh* for generating random field values and (b) *a structural mesh* to carry out all standard FE computations. The stochastic mesh size depends on the correlation length parameters (smaller correlation lengths induce the need for finer meshes), while the structural mesh size is usually determined by the expected gradient of the stress field (steeper stress gradients impose the use of elements with smaller size). As the two meshes typically coincide, the stochastic mesh is usually too fine resulting in severe computational difficulties. However, simulation results of acceptable accuracy are often obtainable with a stochastic mesh, which is considerably coarser than the structural one. Working on a coarser stochastic mesh accelerates random field generation by a factor depending on how much coarser this mesh can be made compared to the structural one.

In this project a new mapping approach between the two types of meshes is introduced. More specifically, random field values are economically generated on a coarser stochastic mesh and then a bivariate interpolation procedure is applied to map these values onto the finer structural mesh. This bivariate interpolation scheme allows the appropriate mapping of random field information between any pair of arbitrarily generated structural and stochastic meshes. Hence, a single mapping concept is capable of treating: (a) stochastic meshes constructed by grouping elements of the structural mesh and (b) completely independent structural and stochastic meshes produced by separately invoking a mesh generator. Using this interpolation algorithm we can effectively deal with irregularly distributed grid points, therefore we can handle structured and unstructured meshes, as well as locally refined meshes with non-uniform element sizes across the discretized domain.

3.2 Computation of stiffness matrices

The TRIC shell element implemented in this project has exhibited satisfactory numerical behavior and computational efficiency. The cost-effectiveness of TRIC ensures the formation of shell stiffness matrices in the context of MCS-based FE analysis in reasonable processing times.

3.3 Solution of finite element equations

In MCS-based stochastic FE analysis successive linear systems with multiple left-hand sides have to be processed, since the coefficient matrix changes in every simulation:

$$K_i u_i = f \Rightarrow (K_0 + \Delta K_i) u_i = f . \quad (1)$$

In the above equation, K_i and u_i are the stiffness matrix and vector of unknown nodal displacements associated with the i th simulation, while K_0 is the stiffness matrix associated with the initial simulation and f is the vector of nodal loads. The difference matrix ΔK_i is generally small compared to K_0 .

The standard direct method based on Cholesky factorization and subsequent forward and backward substitutions remains the most popular solution scheme for FE equations (1). However, this solution approach performs poorly in large-scale problems and the

solution of equations (1) becomes a major computational task that hinders the overall MCS process. Therefore, alternative solution schemes have been sought in this project, in order to handle stochastic FE equations in a computationally more efficient way.

PCG- K_0 is a hybrid method combining both iterative and direct solution concepts. This method uses the iterative Preconditioned Conjugate Gradient (PCG) algorithm equipped with a preconditioner following the rationale of incomplete Cholesky preconditionings. Hence, the incomplete factorization of the stiffness matrix $K_0 + \Delta K_i$ can be written as:

$$\tilde{L}_i \tilde{D}_i \tilde{L}_i^t = K_0 + \Delta K_i - E_i, \quad (2)$$

where \tilde{D}_i is a diagonal matrix, \tilde{L}_i is a lower triangular matrix with unit elements on the leading diagonal and E_i is an error matrix which does not have to be formed. For the typical reanalysis problem (1) matrix E_i is taken as ΔK_i and the preconditioning matrix becomes the complete factorized initial stiffness matrix: $\tilde{K} = K_0$. If matrix ΔK_i is sufficiently small compared to K_0 , we can expect that $\tilde{K} = K_0$ will act as a strong preconditioner for the successive conjugate gradient solutions. The repeated solutions required for the preconditioning step of the PCG algorithm can be efficiently treated as problems with multiple right-hand sides with a direct solution scheme, provided that K_0 is retained in memory in its factorized form.

PCG- K_0 is a customized version of PCG, which takes into account the relatively small differences between successive stochastic stiffness matrices and avoids the treatment of equations (1) as stand-alone problems. This hybrid solution approach runs faster and is less storage demanding than a purely direct solver.

3.4 Parallel execution of the MCS process

Parallel processing is particularly suitable for coping with the excessive computational workloads produced in the context of MCS-based FE analysis. In order to take advantage of high performance computing environments, the global set of simulations to be performed can be decomposed into subsets, each of which is assigned to a different processor. In other words, several simulations are concurrently conducted by executing the standard sequential MCS-based FE procedure on each processor for a part of the total number of simulations. In this project the experimental test bed for distributed computations is a cluster of 16 ethernet-networked Pentium PCs running the Linux operating system and the message passing software Parallel Virtual Machine (PVM). A network-distributed implementation of the MCS procedure is implemented, which allows for inherently parallel computations without any need for inter-processor communication while processing each simulation's FE problem. Although certain tasks of the PCG- K_0 solution method are either of sequential nature or cannot be evenly balanced among the utilized PCs, most major computational tasks of the MCS process are embarrassingly parallel leading to favorable speedups.

3.5 Numerical results

A pinched cylinder is a characteristic test problem, with which the computational procedures described above have been evaluated. The spatial variation of the structure's

modulus of elasticity E and thickness h is represented by two uncorrelated 2D-1V homogeneous Gaussian stochastic fields with coefficients of variation $\sigma_E=\sigma_h=10\%$ and common correlation length $b=2.4\text{m}$. The objective of the MCS-based stochastic analysis of the pinched cylinder problem is to calculate the probability P_f that the absolute value of the structure's vertical displacement at a node exceeds some critical value. A number of TRIC shell FE meshes of various sizes are produced for this stochastic test problem. The finest of these meshes, which consists of 100×35 nodes with 19800 active d.o.f., is adopted as the structural FE mesh. A single stochastic mesh, which may be any of the aforementioned meshes, is employed in each test run of the pinched cylinder example to generate random field values for all uncertain properties involved in the stochastic analysis. The results reported in Table 1 demonstrate the efficiency of the computational approaches introduced in this project.

| Number of PCs | Solution method | Stochastic mesh | Time (s) | | | | | Total storage per PC (MB) | |
|---------------|-----------------|-----------------|-------------------|-------------------------|--------------------|---------------|----------|---------------------------|-------|
| | | | Stochastic fields | Bivariate interpolation | Stiffness matrices | Factorization | Solution | | Total |
| 1 | Direct | 100×35 | 15656.1 | - | 740.1 | 32822.6 | 312.5 | 49531.3 | 57.18 |
| | PCG- K_0 | 100×35 | 15656.1 | - | 518.8 | 6.9 | 475.2 | 16657.0 | 34.60 |
| | PCG- K_0 | 15×5 | 276.4 | 8.1 | 518.8 | 6.9 | 475.3 | 1285.5 | 34.65 |
| 8 | Direct | 100×35 | 1957.0 | - | 92.5 | 4102.8 | 39.1 | 6191.4 | 57.18 |
| | PCG- K_0 | 100×35 | 1957.0 | - | 64.9 | 6.9 | 63.3 | 2092.1 | 34.60 |
| | PCG- K_0 | 15×5 | 34.6 | 1.0 | 64.9 | 6.9 | 63.4 | 170.8 | 34.65 |
| 16 | Direct | 100×35 | 978.5 | - | 46.3 | 2051.4 | 19.5 | 3095.7 | 57.18 |
| | PCG- K_0 | 100×35 | 978.5 | - | 32.4 | 6.9 | 33.1 | 1050.9 | 34.60 |
| | PCG- K_0 | 15×5 | 17.3 | 0.5 | 32.4 | 6.9 | 33.1 | 90.2 | 34.65 |

Table 1. Pinched cylinder: Time allocation and storage requirements (2400 simulations).
Note: practically the same P_f -result is obtained in all MCS runs.

4. Concluding remarks

In this project, a stochastic FE analysis of shell structures with combined uncertain material and geometric properties is performed with the TRIC shell element. In an effort to accelerate the most time consuming tasks performed in MCS-based stochastic shell FE analysis, the drawbacks of standard computational techniques employed to perform such analyses are addressed (coinciding structural and stochastic meshes, direct equation solving, sequential processing) and more efficient alternative procedures are presented (use of coarse stochastic meshes, PCG- K_0 solution method, parallel processing). The effectiveness of these alternatives becomes evident by comparing their computational efficiency to that yielded by conventional techniques. For instance, the total processing time needed to conduct 2400 simulations for a stochastic configuration of the pinched cylinder test problem is reduced from almost 14 hours using standard techniques to just 90 seconds with the aforementioned alternative computational schemes. Thus, the adoption of efficient computational approaches ensures that MCS-results in the context of stochastic shell FE analysis are obtainable in affordable processing times.

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