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A Direct Method for the Plastic Analysis of Structures Under Cyclic Loading

Research Team

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1. Introduction

Structures such as nuclear reactors, aircraft gas turbine propulsion engines, etc., operate in high levels of loads and temperature. It is essential, therefore, for the design of these structures to predict the inevitable accumulation of inelastic strains throughout their life.

The complete response of a structure, which is subjected to a given mechanical loading and exhibits inelastic time independent (plastic) and inelastic time dependent (creep) behaviour, is quite complex. The reasons of complexity are the laborious and often numerically unstable time stepping calculations that have to be performed following the exact loading history. If one is only interested to get an estimation of the strength of the structure, direct methods provide a much better alternative. These methods seek the steady-state solution right from the start of the calculations. Among these methods are the limit and shakedown analysis of structures.

In this work a direct method, already used for creep (Spiliopoulos [1]), is being developed for the plastic cyclic loading analysis of structures.

2. Existing Direct Methods

The first part of the work consists of the examination and extensions of existing direct methods. These methods are based on the upper bound theorem of the theory of plasticity. Two classes of methods are examined: those based on mathematical programming formulations and those based on an iterative method that continuously updates the modulus of elasticity.

2.1 Methods based on mathematical programming

A structure, e.g. a plane structure is divided into an adequate number of elements, whose common edges may serve as possible yield lines of the structure. Every possible collapse mechanism of the structure may be described by the velocity rate of the horizontal and perpendicular component of displacement on the yield line (Spiliopoulos and Politis [2]). The collapse mechanism is kinematically acceptable if the adjacent to the yield line elements may be separated either tangentially or perpendicularly or finally by a combination of both. The perpendicular and tangential relative velocity between elements i and j which are separated by the line *m* are described through the displacement rate of the two elements $\dot{u}_i$, $\dot{v}_i$, $\dot{u}_j$ and $\dot{v}_j$ and are given by the following relation:

$$\dot{e}_n = -(\dot{u}_j - \dot{u}_i)\cdot\sin\alpha + (\dot{v}_j - \dot{v}_i)\cdot\cos\alpha \quad (1)$$

$$\dot{e}_t = (\dot{u}_j - \dot{u}_i)\cdot\cos\alpha + (\dot{v}_j - \dot{v}_i)\cdot\sin\alpha \quad (2)$$

α is the angle between the horizontal principle axis and the common edge, (Fig. 1).

Grouping all the above kinematical equations for all the elements which constitute the structure we may obtain the following matrix equations:

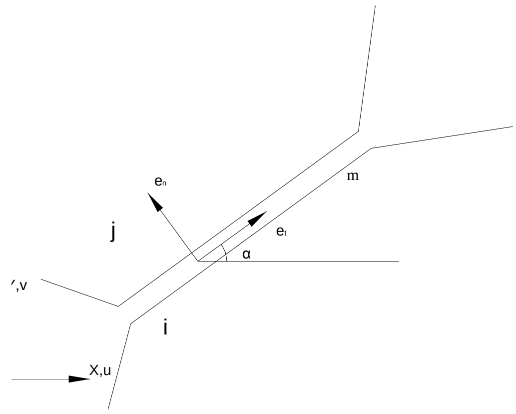

Figure 1. Plasticity lumped along common edges.

$$\dot{\mathbf{e}}_n = \mathbf{B}_n \cdot \dot{\mathbf{d}} \quad (3)$$

$$\dot{\mathbf{e}}_t = \mathbf{B}_t \cdot \dot{\mathbf{d}} \quad (4)$$

The vectors $\dot{\mathbf{e}}_n$ and $\dot{\mathbf{e}}_t$ contain the vertical and tangential rates of the relative displacements. The vector $\dot{\mathbf{d}}$ contains the rate of the displacements.

Along the plasticity line *m* the von Mises yield criterion as expressed for plane structures is used:

$$F = \sigma_m^2 + 3\cdot\tau_m^2 = \sigma_y{}^2 \quad (5)$$

where σ_m is the normal stress, τ_m is the shearing stress along a yield line and σ_y is the yield stress of the material.

If we replace $\sigma_m = X$, $\sqrt{3}\cdot\tau_m = Y$ and $\sigma_y = R$ the above equation is transformed to an equation of a circle whose center lies at the start of the principal axes σ_m and τ_m. Instead of using the full circle one may use the circumscribed polygon. It can be easily shown that the linearized yield criterion can be expressed by the following equation:

$$F_k = \cos a_k \cdot \sigma_m + \sqrt{3}\cdot\sin a_k \cdot \tau_m = \sigma_y \quad (6)$$

with $a_k = 2\cdot k\cdot\pi$, $k = 1, 2, ..., K$, $\beta = \pi/K$, K is the total number of the sides of the circumscribed polygon.

In the case of a *linearized yield criterion* the components of the plastic flow rate on the sides of the linearized equation are obtained by differentiating the above equation:

$$\dot{e}_{n,m} = \dot{\lambda}_m^k \cdot \frac{\partial F_k}{\partial \sigma_m} = \dot{\lambda}_m^k \cdot \cos a_k\,,\ \dot{e}_{t,m} = \dot{\lambda}_m^k \cdot \frac{\partial F_k}{\partial \tau_m} = \sqrt{3} \cdot \dot{\lambda}_m^k \cdot \sin a_k \qquad (7)$$

A straight derivation by differentiating (5) is obtained when maintaining the non-linear criterion in its proper form:

$$\dot{e}_{n,m} = \dot{\lambda}_m \cdot \frac{\partial F}{\partial \sigma_m} = 2 \cdot \dot{\lambda}_m \cdot \sigma_m\,,\ \dot{e}_{t,m} = \dot{\lambda}_m \cdot \frac{\partial F}{\partial \tau_m} = 6 \cdot \dot{\lambda}_m \cdot \tau_m \qquad (8)$$

2.1.1 Governing equations

The rate of the work produced by the external forces which act on the structure is :

$$\dot{W} = \mu \cdot \bar{\mathbf{f}}^T \cdot \dot{\mathbf{d}} \qquad (9)$$

where μ is the load factor and $\bar{\mathbf{f}}$ is the vector of the external loads.

The rate of dissipation of the plastic work along the common lines of adjacent elements is given by the relation:

$$\dot{D} = \sum_{m=1}^{M} \int_0^{l_m} (\sigma_m \cdot \dot{\varepsilon}_m + \tau_m \cdot \dot{\gamma}_m) dl_m = \sum_{m=1}^{M} \int_0^{l_m} (\sigma_m \cdot \dot{e}_{n,m} + \tau_m \cdot \dot{e}_{t,m}) dl_m \qquad (10)$$

where l_m denotes the length of a common edge, M is the total number of the common edges. With either a linearized yield criterion or a non-linearized one, the above equation becomes (11) or (12) respectively:

$$\dot{D} = \sum_{m=1}^{M} \int_0^{l_m} (\sigma_m \cdot \dot{e}_{n,m} + \tau_m \cdot \dot{e}_{t,m}) dl_m = \sigma_y \cdot \sum_{k=1}^{K} \sum_{m=1}^{M} \dot{\lambda}_m^k \cdot l_m \qquad (11)$$

$$\dot{D} = \sum_{m=1}^{M} \int_0^{l_m} (\sigma_m \cdot \dot{e}_{n,m} + \tau_m \cdot \dot{e}_{t,m}) dl_m = 2 \cdot \sigma_y^{\ 2} \cdot \sum_{m=1}^{M} \dot{\lambda}_m \cdot l_m \qquad (12)$$

Equating (9) to either (11) or (12) and requiring the extra constraint $\bar{\mathbf{f}}^T \cdot \dot{\mathbf{d}} = 1$, so that plastic mechanisms may exist, the problem is converted to either a linear (13) or a nonlinear (14) mathematical programming problem:

$$\begin{aligned} &\text{Min } \mu = \sigma_y \cdot \sum_{k=1}^{K} \sum_{m=1}^{M} \dot{\lambda}_m^k \cdot l_m \\ &\text{Subject to } \dot{\mathbf{e}}_n - \dot{\mathbf{B}}_n \cdot \dot{\mathbf{d}} = \mathbf{0} \\ &\qquad \dot{\mathbf{e}}_t - \dot{\mathbf{B}}_t \cdot \dot{\mathbf{d}} = \mathbf{0} \\ &\qquad \bar{\mathbf{f}}^T \cdot \dot{\mathbf{d}} = 1 \\ &\qquad \dot{\lambda}_m^k \geq 0 \end{aligned} \qquad (13)$$

$$\begin{aligned} &\text{Min } \mu = 2 \cdot \sigma_y^{\ 2} \cdot \sum_{m=1}^{M} \dot{\lambda}_m \cdot l_m \\ &\text{Subject to } \dot{\mathbf{e}}_n - \dot{\mathbf{B}}_n \cdot \dot{\mathbf{d}} = \mathbf{0} \\ &\qquad \dot{\mathbf{e}}_t - \dot{\mathbf{B}}_t \cdot \dot{\mathbf{d}} = \mathbf{0} \\ &\qquad \sigma_m^{\ 2} + 3 \cdot \tau_m^{\ 2} = \sigma_y^{\ 2} \\ &\qquad \bar{\mathbf{f}}^T \cdot \dot{\mathbf{d}} = 1 \\ &\qquad \dot{\lambda}_m \geq 0 \end{aligned} \qquad (14)$$

2.1.2 Solution of the mathematical programs

A critical issue is the amount of computing time for the solution of any of the two programs described above. Non-linear optimization techniques are used in this work (Corn et al. [3]), even for the linear program, since when using a good starting solution we get a quick convergence. In an alternative solution method for the linear programming problem (13), the simplex method, despite its convergence in a finite number of steps, a lot of extra (artificial) variables must be introduced, for its solution, at the expense of extra computing time.

2.2 Methods Using Linear Elastic Solutions With a Spatially Varying Elastic Modulus

These methods (Ponter and Carter [4]) are also based on the kinematical theorem of plasticity and produce a sequence of lower bounds of increasing accuracy. The main essence of these methods is to adjust the elastic modules within a finite element scheme so that the stresses are brought within the yield condition at a fixed strain distribution. The elastic problem is then resolved using the new spatial distribution of elastic moduli. At each stage, a lower bound on the limit load can be found by scaling the solutions so that the stresses lie within yield for the current elastic solution. Experience has shown that a monotonically increasing sequence of lower bounds is usually obtained. The steps of this method may be roughly summarized as follows:

1. The structure is discretized and all boundary conditions and external loading are applied to it. The elastic properties of the material are used except for the Poisson's ratio which is set to a value near 0.5 so that the material is incompressible.
2. The structure is then solved elastically and the stresses at each element's Gauss point are computed. The values of internal and external work are calculated.
3. Young's modulus is updated at each element's Gauss point according to relation (15) and recalculation of the structure's stiffness matrix takes place.

$$E^{k+1} = a \cdot E^{k} \cdot \frac{\sigma_y}{\overline{\sigma}(\tilde{\sigma}_{ij}^{k})} \tag{15}$$

4. Compute the rate of internal work in respect of external work (21).

$$P_{UB}^{k} = \frac{\int_V \sigma_{ij}^{\prime c} \cdot \tilde{\varepsilon}_{ij}^{k} \cdot dV}{\int_S \overline{p}_i \cdot u_i^{k} \cdot dS} \Leftrightarrow P_{UB}^{k} = \sqrt{\frac{2}{3}} \cdot \sigma_y \cdot \frac{\int_V \sqrt{\tilde{\varepsilon}_{ij}^{k} \cdot \tilde{\varepsilon}_{ij}^{k}} \cdot dV}{\int_S \overline{p}_i \cdot u_i^{k} \cdot dS} \tag{16}$$

5. Check if tolerance is acceptable or else return to 2nd step.

where $\overline{\sigma}(\tilde{\sigma}_{ij}^{k})$ is an equivalent stress value that may or may not exceed the yield stress.

3. Numerical example-limit analysis

The procedure is applied to the limit analysis of a square plate with a circular hole of diameter of 1/10 of the side of the square.The dimensions of the plate are 20m x 20 m x 1m. Two uniformly distributed loads parallel to the horizontal and to the vertical axes having a maximum value equal to $P1=P2=\sigma_y$ are applied at the far ends of the plate. Due to the symmetry of the problem, only a quarter of the plate was discretized with 98 quadrilateral elements (Fig.2). The sequence of loading is the following: First P1 is applied as a whole with its maximum value. After that, loading P2 is augmented every time by 10% of the maximum value until it also reaches its maximum value. P1 is then decreased by portions of 10% of its maximum value until it is zeroed.

It can be seen from Fig.3 that there is little difference in the results of the two programs (13) and (14). It must be noted, nevertheless that the non-linear program (14) uses fewer amounts of variables and constraints and therefore needs less computing time. The program (13), however, seems to converge from any starting point (e.g. null variables). This is not the case for the program (14) for which some of the variables may be chosen so as to satisfy the yield constraint in an arbitrary way, i.e. all the normal stresses are put equal to σ_y and all the other variables are zeroed.

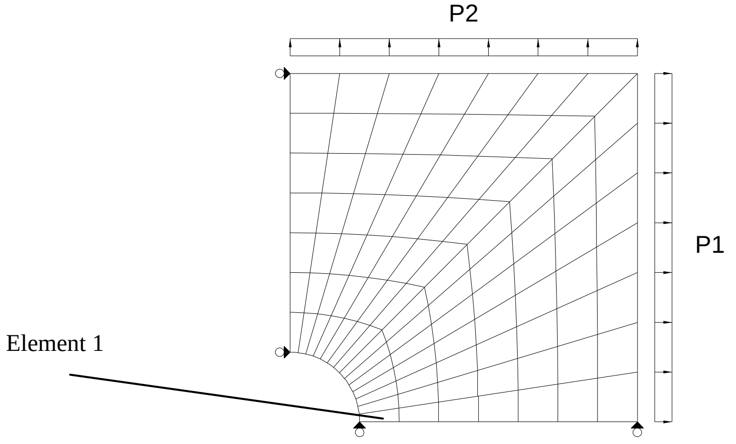

Figure 2. Finite element discretization of the plate.

Results were compared (Fig. 3) against a time-stepping program (Hibbit et al. [5]) which uses an arc-length method. The same discretization was employed for both methods but the running time was approximately 80 times less for the direct methods. The difference in the value of the limit load as compared with the one of the time-stepping procedure varied between 3.7% and 12.0%. This discrepancy is expected due to the fact the direct method pre-assumes a collapse mode along the edges of the elements whereas the time-stepping method takes into account plastification inside the elements. This discrepancy is expected to decrease with more refined discretization.

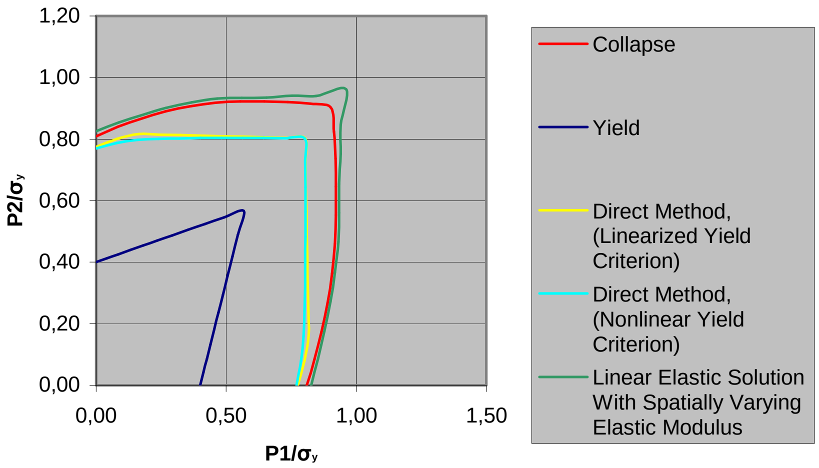

Figure 3. Results for the plate problem.

4. Numerical example-cyclic loading

The previous problem was solved under the cyclic variation of the two end loads that appear in Fig. 4, using Hibbit et al. [5]. Steady state results of equivalent plastic strain can be seen in Fig.4. Results of this analysis are expected to be compared to the ones using the method under development, which is presented in the next section.

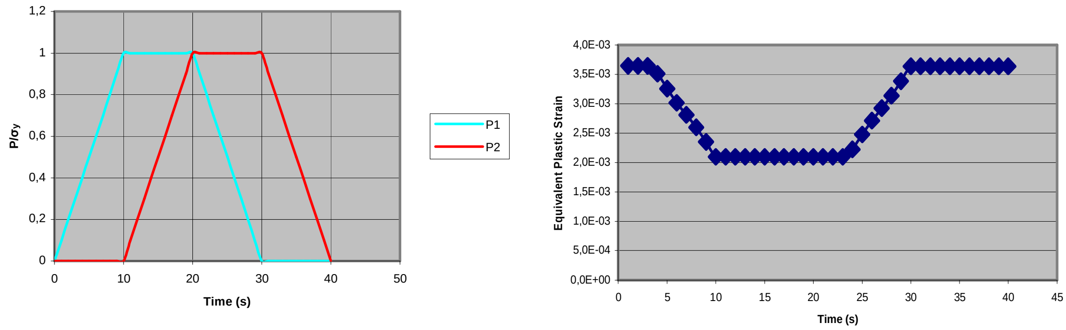

Figure 4. (a) Loading variation through time, (b) Equiv. plast. strain at elem. 1 at Gauss point 1

5. Direct method using Fourier series

A direct method (Spiliopoulos [1]) that may be used in cyclic loading analysis is proposed, which currently is under testing. The method is based on the Fourier decomposition of the residual stresses which develop under plastic behaviour and are expected to be also cyclic:

$$\boldsymbol{\rho}(t) = \frac{\mathbf{a}_0}{2} + \sum_{k=1}^{N}\left(\boldsymbol{\alpha}_k \cos\frac{2k\pi t}{T} + \mathbf{b}_k \sin\frac{2k\pi t}{T}\right) \tag{17}$$

where T is the period of the cycle.

The various coefficients of the Fourier series may be found in an iterative manner by satisfying equilibrium and compatibility at time points inside the cycle. It turns out that if both conditions are satisfied at a time point inside a cycle, the time derivative of the residual stress may be calculated from the following equations:

$$\mathbf{K}\dot{\mathbf{r}} = \dot{\mathbf{R}} + \int_V \mathbf{B}'\mathbf{D}\dot{\boldsymbol{\varepsilon}}^{pl}dV \tag{18}$$

$$\dot{\boldsymbol{\rho}} = \mathbf{D}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{el} - \dot{\boldsymbol{\varepsilon}}^{pl}) \tag{19}$$

where $\mathbf{K}$ is the stiffness matrix of the structure, $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^{pl}$ are the total and the plastic strains respectively, $\dot{\boldsymbol{\varepsilon}}^{el}$ are the strains assuming elastic behaviour ,and $\dot{\mathbf{R}}, \dot{\mathbf{r}}$ are the rates of loading and of the residual displacements respectively.

6. References

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