Mathematical Analysis of Some Nonlinear Differential Equations

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Program **Thalis No 65/1324**

A. C. V. Nikolopoulos and D. E. Tzanetis, *A model for housing allocation of a homeless population due to a natural disaster*, Nonlinear Anal. Real World Appl. 4 (2003), no. 4, 561–579.

In the present work we derive and analyze a model considering housing allocation of homeless families due to a natural disaster. We use the data from the earthquake of September, 1999 in Athens, Greece. We derive a non-linear system of ordinary differential equations. We analyze the stability of the system and find an approximate solution of the model for a case study as well as and a numerical solution. Finally we consider possible extensions and improvements of the model making it more realistic.

Main Results

Making some simplifications of the physical problem we first construct a model consists of a system of two ordinary differential equations and an algebraic one for $t \geq 0$:

$$\frac{dW}{dt} = -k_1 W(Q_a - Q) - k_2 W(R_a - R), \tag{0.1}$$

$$\frac{dQ}{dt} = k_1 W(Q_a - Q) - k_3 Q(R_a - R), \tag{0.2}$$

$$W_0 = W(0) = W(t) + Q(t) + R(t), \tag{0.3}$$

whrere $W = W(t), Q = Q(t)$ and $R = R(t)$ stand for the number of families having houses destroyed or badly damaged, the number of families are accommodated in a temporary state and the number of families that are resettled in a new home respectively. At the first stage using some data we compute the constants $k's$ and then using center manifolds arguments we study the stability of the equillibrium points of the system.

Under some different hypotheses we construct the more sophisticated model

$$\frac{dW}{dt} = -K_1W(T_a - T) - K_2W - K_7W(R_a - R), \tag{0.4}$$
$$\frac{dT}{dt} = K_1W(T_a - T) - K_5T(R_a - R) - K_3T(P_a - P), \tag{0.5}$$
$$\frac{dP}{dt} = K_3T(P_a - P) - K_4P(R_a - R), \tag{0.6}$$
$$\frac{dF}{dt} = -K_6F(R_a - R) + K_2W, \tag{0.7}$$
$$W_0 = W(0) = W(t) + R(t) + P(t) + T(t) + F(t), \tag{0.8}$$

where $W = W(t), T = T(t), P = P(t), F = F(t)$ and $R = R(t)$ stand for the number of families that become homeless after the natural disaster, the number of families living in house in camp organized by the state, the number of families living with relatives or friends and the number of families that are resettled respictively. Again, the stability of stationary points is studied.

B. N. I. Kavallaris and D. E. Tzanetis, *Behaviour of a non-local reactive-convective problem with variable velocity in Ohmic heating of food*, Banach Center Publications 66, 189-198 (2004).

We consider the non-local equation

$$\rho(u)u_t + \left(1 - \int_0^x \rho'(u)u_t dy\right) u_x = \frac{\lambda f(u)}{\left(\int_0^1 f(u)dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{0.9}$$

$$u(0,t) = 0, \quad t > 0, \tag{0.10}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{0.11}$$

which models the temperature when an electric current flows through a moving material with negligible thermal conductivity and time-dependent velocity. The potential difference across the material is fixed but the electrical resistivity $f(u)$ and the density $\rho(u)$ vary significantly with temperature. It is found that for $f(u)$ decreasing with $\int_0^\infty f(s)ds < \infty$, blow-up occurs if λ is too large for a steady state to exist or if the initial condition is large enough. On the other hand, if $f(u)$ is decreasing with $\int_0^\infty f(s)\,ds = \infty$, then it is proved that $u(x,t)$ is a global-in-time and unbounded solution.

Main Results

Proposition 0.1 *If* $\int_0^\infty f(s)ds < \infty$ *and* $\rho(s) \geq \gamma > 0$ *for* $s > 0$ *then the unbounded solutions to (0.9)-(0.11) blow up globally in finite time, i.e.* $u(x,t) \to \infty$ *as* $t \to t^*- < \infty$ *for any* $x \in (0,1]$ *and* $u_x(0,t) \to \infty$ *as* $t \to t^* -$.

Proposition 0.2 *If* $\int_0^\infty f(s)ds = \infty$ *and* $\rho(s) \geq \gamma > 0$ *for* $s > 0$ *then any unbounded solution* $u(x,t)$ *to (0.9)-(0.11) diverges globally, i.e.* $u(x,t) \to \infty$ *as* $t \to \infty$ *for any* $x \in (0,1]$ *and* $u_x(0,t) \to \infty$ *as* $t \to \infty$.

C. C.V Nikolopoulos, *A model for melting of an inhomogeneous material during modulated temperature differential scanning calorimetry*, Appl. Math. Modelling 28, No.5, 427-444 (2004).

In the present work a model considering the melting of an inhomogeneous material, such as a mixture or solution, during Modulated Temperature Differential Scanning Calorimetry (MTDSC), is derived and analysed. It is considered that during the melting of such a material a mushy region is formed and initially the behaviour of the material at the microscale is analysed. Then with the method of averaging a phase field type model is constructed for the macroscale which consists of a system of partial differential equations and this is solved numerically. Finally the results are used to simulate the signal of MTDSC.

Main Results

A representation for the melting of an inhomogeneous material can be made by constructing a model which takes into account the formation of a mushy region inside the sample during the melting. The sample will be taken to be a slab (such as a thin disk) and essentially one dimensional. The disc is symmetric so we need to study only one half of it. The surfaces of the disk are taken to have temperature T_s controlled in the usual way, $T_s = T_o + bt + B\sin(\omega t)$. Considering now the melting process inside the sample we will assume that the part of the mushy region which is near a purely liquid zone will consist of small shrinking solid spheres imbedded in liquid and that near a purely solid zone we have small liquid spheres growing in solid. Finally the phase field model derived has the form of a system of equations for the macroscopic temperature θ and the solid fraction α.

The temperature field will be described by the system of equations

$$\varepsilon\frac{\partial\theta}{\partial\tau} = \nabla^2\theta(y,\tau) - C_m f, \tag{0.12}$$

$$\text{where} \qquad \frac{\partial\alpha}{\partial\tau} = -f. \tag{0.13}$$

The function f is defined to be:

$$f = \begin{cases} 0 \quad \text{in the purely liquid zone,} \\ C_n\theta(y,\tau)\left[(\alpha_c-\alpha)\alpha\right]^{\frac{1}{3}} - C_k(2\alpha-1) \quad \text{in the mushy region,} \\ 0 \quad \text{in the purely solid zone.} \end{cases} \tag{0.14}$$

We can approximate the solution of (0.12) with an explicit finite difference scheme.

$$\theta_j^{n+1} = \theta_j^n + \frac{\delta\tau}{\varepsilon\delta y^2}\left(\theta_{j-1}^n - 2\theta_j^n + \theta_{j+1}^n\right) - \delta\tau\left(f_1(\alpha_j^n)\theta_j^n - f_2(\alpha_j^n)\right).$$

We pose the same boundary conditions as before, i.e. $\theta(0,\tau) = C_m\tau$ and $\frac{\partial\theta}{\partial y}(1,\tau) = 0$, and for initial condition $\theta(y,0) = \frac{\varepsilon}{2}\left((1-y)^2-1\right)$. The latter comes from solving the heat equation. We assume that temperature T, before melting starts, has the form $T(x,t) = \bar{T}(x) + bt + BIm\{\hat{T}(x)e^{i\omega t}\}$ for $t<0$ (while the sample remains solid). This gives that for $t=0$, $T(x)$ is approximately $T(x,0) \simeq -\frac{\rho cb}{2k}\left[L_s^2 - (L_s-x)^2\right]$. In this way we can obtain a numerical approximation for the underlying part of θ while at each time step α can be approximated by the equation

$$\alpha_j^{n+1} = \alpha_j^n + \delta\tau\left\{\left[-\theta_j^n C_n(\alpha_j^n(\alpha_c-\alpha_j^n))^{\frac{1}{3}}\right] + C_k(2\alpha_j^n-1)\right\}. \tag{0.15}$$

The effect of keeping the time derivative in equation (0.12) is that the material needs more time to melt due to the heat absorption related with the heat capacity of the sample.

As regards the modulation, the resulting equation for the amplitude of temperature modulation is

$$\frac{\partial^2\hat{\theta}(y,\tau)}{\partial y^2} + i\omega_0\left(\frac{C_m f_1(\bar{\alpha})}{\left[\left(\frac{df1}{d\alpha}\right)_{\alpha}\bar{\theta} - C_k^*\right] + i\omega_0} + \varepsilon\right)\hat{\theta} = 0, \tag{0.16}$$

with $\hat{\theta}=0$, at $y=0$ and $\frac{\partial\hat{\theta}}{\partial y}=0$, at $y=1$, while the equation for the amplitude of the solid fraction $\hat{\alpha}$ is given again by equation

$$\hat{\alpha}(y,\tau) = \frac{-f_1(\bar{\alpha})\hat{\theta}(y,\tau)}{\left[\left(\frac{df_1}{d\alpha}\right)_{\bar{\alpha}}\bar{\theta} + 2C_k^*\right] + iC_m\omega_0}. \tag{0.17}$$

Note that for large frequency, i.e. $\omega_0 \gg 1$ and with $\varepsilon \sim O(1)$, we have that the term multiplying $\hat{\theta}$ can be approximated by $i\omega_0\varepsilon$ so equation (0.16) becomes

$$\frac{\partial^2\hat{\theta}(y,\tau)}{\partial y^2} + i\omega_0\varepsilon\hat{\theta} = 0. \tag{0.18}$$

This gives $\hat{\theta} = e^{-(i+1)\sqrt{\frac{\omega_0\varepsilon}{2}}}\sinh\left[(i+1)\sqrt{\frac{\omega_0\varepsilon}{2}}y\right]/\cosh\left[(i+1)\sqrt{\frac{\omega_0\varepsilon}{2}}\right] + e^{-(i+1)\sqrt{\frac{\omega_0\varepsilon}{2}}y}$. Using the numerical approximation for $\bar{\theta}$ and this expression for $\hat{\theta}$, a simulation of the MTDSC measurements is possible

Figure 1: Simulation of underlying (solid line) and cyclic (dotted line), the form of MTDSC measurements of heat capacity, for the case that the melting temperature is given by the Gibbs-Thompson condition.

D. C. V. Nikolopoulos and D. E. Tzanetis, *Blow-up time estimates for a non-local reactive-convective problem modelling sterilization of food*, Banach Center Publications 66, 237-198 (2004)

We consider the non-local initial boundary value problem,

$$u_t(x,t) + u_x(x,t) = \lambda \frac{f(u(x,t))}{\left(\int_0^1 f(u(x,t))\,dx\right)^2}, \qquad 0 < x < 1, \quad t > 0, \tag{0.19}$$

$$u(0,t) = 0\,, \;\; t > 0, \tag{0.20}$$

$$u(x,0) = u_0(x) \geq 0, \qquad 0 < x < 1, \tag{0.21}$$

where $\lambda > 0$. The function $u(x,t)$ represents the dimensionless temperature when an electric current flows through a conductor (e.g. food) with temperature dependent on electrical resistivity $f(u) > 0$, subject to a fixed potential difference $V > 0$. The (dimensionless) resistivity $f(u)$ may be either an increasing or a decreasing function of temperature depending strongly on the type of the material (food). Problem (0.19)-(0.21) models one of the main methods for sterilizing food. The sterilization can take place by electrically heating the food rapidly.

Main Results

Estimates of the blow-up time t^* of the form

$$t_l(\lambda - \lambda^*)^{-1/2} \leq t^* \leq c_1 + c_2 \ln\left[c_3\,(\lambda - \lambda^*)^{-1}\right],$$

where $t_l = c_4\,\lambda^{-1/2}$ for some positive constants $c_i, i = 1, 2, 3, 4$ are obtained by using comparison methods. Also an asymptotic analysis is applied when $f(s) = e^s$, for $\lambda - \lambda^* \ll 1$, regarding the form of solution during blow up and an asymptotic estimate of blow up time of the form

$$t^* \sim \hat{t}\,(\lambda - \lambda^*)^{-1/2}, \tag{0.22}$$

where $\hat{t} = \pi\,(4\,B\,K)^{-1/2}$ for some positive constants K, B is obtained. Finally some numerical results are also presented.

E. C. V. Nikolopoulos and D. E. Tzanetis, *Estimates of blow-up time of a non-local reactive-convective problem modelling Ohmic heating of foods*, Proceedings of Edinburgh Mathematical Society, (2005), to appear.

Main Results

In this work, we estimate the blow-up time for the non-local hyperbolic equation of Ohmic type, $u_t + u_x = \lambda f(u)/(\int_0^1 f(u)\,dx)^2$, together with initial and boundary conditions. It is known, that for $f(s)$, $-f'(s)$ positive and $\int_0^\infty f(s)ds < \infty$, there exists a critical value of the parameter $\lambda > 0$, say λ^*, such that for $\lambda > \lambda^*$ there is no stationary solution and the solution $u(x,t)$ blows up globally in finite time t^*, while for $\lambda \leq \lambda^*$ there exist stationary solutions. Moreover the solution $u(x,t)$ also blows up for large enough initial data and $\lambda \leq \lambda^*$. Thus, estimates for t^* were found either for λ greater than the critical value λ^* and fixed initial data $u_0(x) \geq 0$, or for $u_0(x)$ greater than the greatest steady-state solution (denote $w_2 \geq w^*$) and fixed $\lambda \leq \lambda^*$. The estimates are obtained by comparison, by asymptotical and by numerical methods. Finally, amongst the others, for given λ, λ^* and $0 < \lambda - \lambda^* \ll 1$, estimates of the form were found: upper bound $\epsilon + c_1 \ln\,[c_2\,(\lambda - \lambda^*)^{-1}]$; lower bound $c_3(\lambda - \lambda^*)^{-1/2}$; asymptotic estimate $t^* \sim c_4(\lambda - \lambda^*)^{-1/2}$ for $f(s) = e^{-s}$. Moreover, for $0 < \lambda \leq \lambda^*$ and given initial data $u_0(x)$ greater than the greatest steady-state solution $w_2(x)$, we have upper estimates: either $c_5 \ln(c_6\,A_0^{-1} + 1)$ or $\epsilon + c_7 \ln(c_8 \zeta^{-1})$, where A_0, ζ measure, in some sense, the difference $u_0 - w_2$, (if $u_0 \to w_2+$ then A_0, $\zeta \to 0+$). $c_i > 0$ are some constants and $0 < \epsilon \ll 1$, $0 < A_0$, ζ. Some numerical results are also given.

F. N. I. Kavallaris, A. A. Lacey, C. V. Nikolopoulos and and D .E. Tzanetis, *Asymptotic analysis and estimates of blow-up time for the radial symmetric semilinear heat equation in the "open-spectrum" case*, submited

We estimate the blow-up time for the reaction diffusion problem

$$u_t(x,t) = \Delta u(x,t) + \lambda f(u(x,t)), \quad x \in \Omega \subseteq R^N, \quad t > 0, \tag{0.23}$$

$$u(x,t) = 0\,,\; x \in \partial\Omega \quad t > 0, \tag{0.24}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{0.25}$$

in the case where f is a positive, increasing and convex function growing fast enough and $\lambda > \lambda^*$, where λ^* is a critical value for the parameter λ regarding the existence of solutions to the corresponding steady-state problem.

<u>Main Results</u>

Estimates of the blow-up time t^* of the form

$$t_l(\lambda - \lambda^*)^{-1/2} \leq t^* \leq t_u(\lambda - \lambda^*)^{-1/2},$$

where $t_l = \frac{1}{2\sqrt{2}}\left(\frac{K}{I}\right)^{1/2}$ and $t_u = \pi(I_1\,\Lambda)^{-1/2}$, for some constants K, Λ, I, I_1, are obtained by using comparison methods. Also an asymptotic analysis is applied when $f(s) = e^s$, for $\lambda - \lambda^* \ll 1$, regarding the form of solution during blow up and an asymptotic estimate of blow up time of the form

$$t^* \sim K_1 \ln\left(K_2 \lambda^* (\lambda - \lambda^*)^{-1}\right). \tag{0.26}$$

for some positive constants K_1, K_2, is obtained. Finally some numerical results are also presented.