

DYNAMICS OF COUPLED OSCILLATORS WITH ATTACHED NONLINEAR ENERGY SINKS

Research Team

Alexander F. Vakakis (*Professor*),
Stylianos Tsakirtzis, Ioanna Rizou (*Research Students*),
School of Applied Mathematical and Physical Sciences

Introduction

We study resonance capture phenomena leading to energy pumping in systems with multiple degrees-of-freedom (DOF), composed of N linear oscillators weakly coupled to strongly nonlinear attachments possessing essential (nonlinearizable) cubic stiffness nonlinearities. First we present numerical evidence of energy pumping in the systems under consideration, i.e., of passive, one-way (irreversible) transfer of externally imparted energy to the nonlinear attachments, provided that the energy is above a critical level. By introducing an approximation based on Jacobian elliptic functions we derive an approximate set of two nonlinear integro-differential modulation equations that govern the time evolution of the amplitude and phase of the motion of the attachment. In addition, we study the possibility of multi-frequency nonlinear energy pumping simultaneously from multiple linear modes through the use of multi-DOF nonlinear attachments.

2. Formulation of the Problem and Numerical Evidence

The system under consideration is a finite chain of N particles with linear grounding stiffnesses, undergoing linear next-neighbor interactions. The chain is coupled at its right boundary to a strongly nonlinear, weakly damped oscillator (attachment). We wish to study the nonlinear interaction of the chain with the attachment, and, in particular, nonlinear energy transfer exchanges resulting from this interaction. The set of equations governing the dynamics is as follows,

$$\begin{aligned} \ddot{u}_1 + u_1(\omega_0^2 + 2\alpha) - \alpha u_2 &= 0 \\ \ddot{u}_n + u_n(\omega_0^2 + 2\alpha) - \alpha(u_{n+1} - u_{n-1}) &= 0, \quad n = 2, 3, \dots, N-1 \\ \ddot{u}_N + u_N(\omega_0^2 + \alpha + \varepsilon) - \alpha u_{N-1} - \varepsilon v &= 0 \\ \ddot{v} + C v^3 + \varepsilon \beta \dot{v} + \varepsilon(v - u_N) &= 0 \end{aligned} \tag{1}$$

where u_n denotes the displacement of the n -th particle of the chain, v the displacement of the nonlinear oscillator, α the coupling between particles of the chain, β the damping coefficient of the nonlinear oscillator, and ω_0^2 the stiffness of the on-site (grounding) quadratic potential. The perturbation parameter $0 \ll \varepsilon \ll 1$ scales the weak coupling between the chain and the nonlinear oscillator, and the parameter C denotes the strength of the essential (nonlinearizable) stiffness nonlinearity. As usual, dot denotes differentiation with respect to time, and the particles are assumed to be of unit mass. To analyze the nonlinear interaction between the attachment and the chain in (1) it is instructive to initially compute the approximate instantaneous frequency of the nonlinear oscillator during the motion [1].

A numerical computation based on a two-DOF chain ($N=2$) demonstrates some important issues of the chain-attachment interaction. In Fig. 1 we depict the transient responses and the approximate instantaneous frequency $\Omega(t)$ of the nonlinear oscillator for a system with

parameters $\alpha = 1$, $\omega_0 = 1$, $\beta = 2$, $C = 3$ and $\varepsilon = 0.1$. We used zero initial conditions except for $\dot{u}_1(0) = Y$; these initial conditions correspond to impulsive excitation at $t = 0$ of the farthest from the attachment oscillator 1. For $Y = 5.6$ no significant nonlinear interaction between the chain and the attachment takes place, and most of the induced energy of vibration remains in the linear part of the system, where it is originally generated. However, by increasing the initial condition to $Y = 8.6$ strong energy transfer to the nonlinear attachment is observed.

The enhanced nonlinear interaction as energy increases can be better understood by considering the plots of Figure 1d, depicting the variation of the instantaneous frequency $\Omega(t)$ of the nonlinear attachment for the two aforementioned cases of impulsive excitation. Indeed, for the lower level excitation $\Omega(t)$ does not reach the neighbourhood of the natural frequencies of the linear chain, and, as a result no resonance interaction (capture) between the attachment and the chain can occur. By contrast, for the case of higher impulsive excitation the instantaneous frequency of the nonlinear oscillator reaches the neighborhood of the smallest natural frequency ω_1 of the linear chain giving rise to *1:1 resonance capture* [1]. By this we mean the *transient* internal resonance between the attachment and the chain in a small neighborhood of a 1:1 resonant manifold of the dynamics [2-5]. Hence, the nonlinear attachment engages in a 1:1 transient resonance interaction with the lowest mode of the linear chain, during which one-way transfer (pumping) of energy to the attachment takes place.

By increasing the magnitude of the impulse to $Y = 25$, there occurs a resonance capture cascade, whereby the attachment transiently resonates with both modes of the linear chain in sequential order. This can be concluded from the frequency plot of Figure 2, where it is seen that the instantaneous frequency $\Omega(t)$ of the nonlinear oscillator first reaches the neighbourhood of the natural frequency of the higher, anti-phase mode of the linear chain, and then makes the transition to the neighbourhood of the natural frequency of the lower, in-phase mode. Damping dissipation is the mechanism that reduces continuously the overall energy of the system and induces the frequency transitions.

3. Analytical Treatment of Transient Resonance Interactions

We now perform an analytic investigation of the transient resonant interactions of the nonlinear attachment with the modes of the linear chain. First, we consider the chain and consider the force exerted by the coupling stiffness to be a pseudo-forcing term. We then obtain a *single* essentially nonlinear, damped integro-differential equation that governs *exactly* the transient dynamics of the nonlinear attachment:

$$\ddot{v} + Cv^3 + \varepsilon\beta\dot{v} + \varepsilon v = \varepsilon \sum_{i=1}^N \sum_{k=1}^N \phi_{i,N} \phi_{i,k} u_k(0) \cos(\omega_i t) + \varepsilon \sum_{i=1}^N \sum_{k=1}^N \frac{\phi_{i,N} \phi_{i,k} \dot{u}_k(0)}{\omega_i} \sin(\omega_i t) + \varepsilon^2 \sum_{i=1}^N \frac{\phi_{i,N} \phi_{i,N}}{\omega_i} \int_0^t v(\lambda) \sin \omega_i(t - \lambda) d\lambda \quad (2)$$

This equation is solved with initial conditions $v(0)$ and $\dot{v}(0)$. Hence, we have reduced the problem of nonlinear interaction of the chain with the nonlinear attachment to a single integro-differential equation. To analyze resonance capture and energy pumping from the N-DOF chain to the attachment, we transform the nonlinear integro-differential equation (2) into a set of two first order nonlinear integro-differential equations that govern the time evolution of the amplitude and phase of the motion of the attachment. For this purpose we consider (2) to be in the form of a nonhomogeneous nonlinear differential equation in order to apply the method of variation of parameters. We first consider the solution of the homogeneous problem,

$$\ddot{v} + Cv^3 = 0 \Rightarrow v(t) = A \operatorname{cn} \left[A\sqrt{C} t + 4K \left(1/\sqrt{2} \right) \xi; 1/\sqrt{2} \right] \quad (3)$$

where A and ξ denote arbitrary constants, and $K(1/\sqrt{2})$ is the elliptic integral of the first kind.

Applying an ‘elliptic’ version of the method of variation of parameters [5-10], we seek the solution of the inhomogeneous differential equation (10) in the form,

$$v(t) = A(t) \operatorname{cn}(A(t)\sqrt{C}t + 4K\xi(t); k) \quad (4)$$

where $A(t)$ and $\xi(t)$ are new unknown, *slowly varying amplitude and phase functions*, respectively. In writing (4) we partition, in essence, the dynamics into fast and slow varying components, at the original time scale t and at a time scale $\varepsilon^\alpha t$ ($\alpha > 0$ to be determined), respectively. The plan is to average out the fast dynamics, and to confine the analysis to the slowly varying dynamics (the *slow flow*) where information on the attachment – chain nonlinear interaction is contained. The slowly varying amplitude and phase functions are governed by the following set of modulation equations:

$$\begin{aligned} \frac{dA}{dt} &= \varepsilon \frac{cn'}{A\sqrt{C}} f(t; \varepsilon) \\ \frac{d\xi}{dt} &= -\varepsilon \frac{cn}{4KA^2\sqrt{C}} f(t; \varepsilon) \end{aligned} \quad (5)$$

We note that both derivatives are of $O(\varepsilon)$, indicating that indeed the amplitude and the phase are slowly varying functions.

We simplify the problem by using the following Fourier expansion for the elliptic cosine,

$$\operatorname{cn}(2K\theta/\pi) = 0.955 \cos \theta + 0.043 \cos 3\theta + \dots \cong \cos \theta \quad (6)$$

and approximating the response of the nonlinear attachment by,

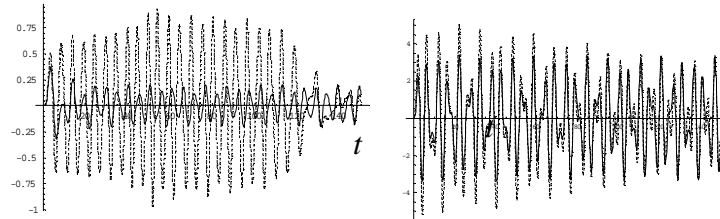
$$v(t) = A(t) \cos \theta(t) \quad (7)$$

where the angle variable is defined as,

$$\theta(t) = \frac{\pi\sqrt{C}}{2K} A(t)t + 2\pi\xi(t)$$

Expressing the modulation equations (5) using the approximation (6) and the new angle variable, we obtain the following approximate modulation equations governing the slow flow of the system:

$$\begin{aligned} \frac{dA}{dt} &= -\varepsilon \frac{\pi \sin \theta(t)}{2KA(t)\sqrt{C}} f(t; \varepsilon) \\ \frac{d\theta}{dt} &= \frac{\pi\sqrt{C}}{2K} A(t) - \varepsilon \frac{\pi \cos \theta(t)}{2K(A(t))^2\sqrt{C}} f(t; \varepsilon) \end{aligned} \quad (8)$$



(a)

(b)

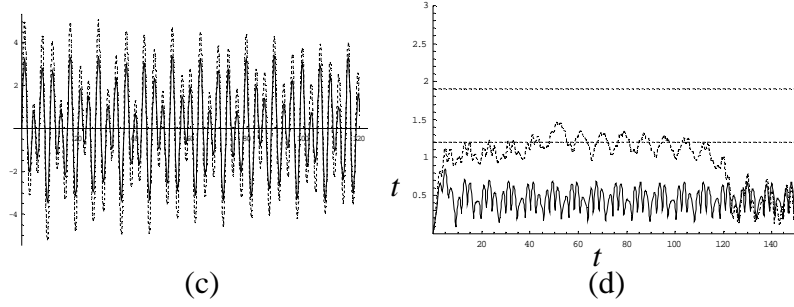


Figure 1. Numerical transient responses of the two-DOF system with nonlinear attachment: (a) $v(t)$, (b) $u_2(t)$, (c) $u_1(t)$, (d) Instantaneous frequency $\Omega(t)$; ——— $Y = 5.6$; - - - - - $Y = 8.6$.

From (8) it directly follows that $\frac{dA}{dt}$ is of order $O(\varepsilon)$ and $\frac{d\theta}{dt}$ is of order $O(1)$. The modulation equations approximately govern the amplitude and phase of the nonlinear attachment as it interacts with the modes of the linear chain. We stress that since we omitted terms of $O(\varepsilon^3)$ from (8), it is expected that their (transient) solutions will be valid only up to times of $O(1/\varepsilon^2)$. We first establish the accuracy of the approximate modulation equations by computing the response of the attachment by (8), and comparing it to direct numerical simulations of the original equations of motion (1). For this comparison we employ a two-DOF linear chain ($N = 2$) with a nonlinear attachment at its end, and parameters $\alpha = 1$, $\omega_0 = 1$, $\beta = 2$, $C = 3$, $\varepsilon = 0.1$; all initial conditions were taken as zero, with the exception $\dot{u}_1(0) = u$.

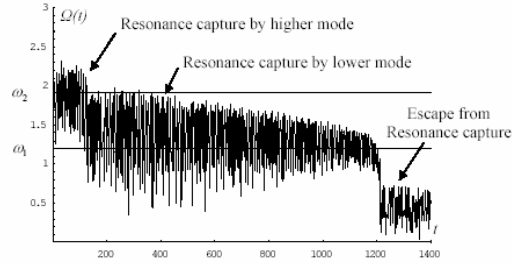


Figure 2. Instantaneous frequency $\Omega(t)$ of the two-DOF system for $Y = 25$.

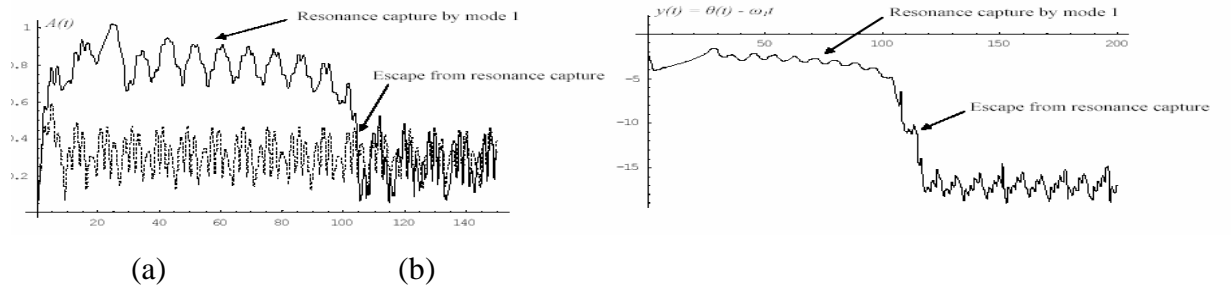


Figure 3. Approximate transient response of the nonlinear attachment: (a) Amplitude $A(t)$ for initial conditions $u = 5.6$ - - - - - , and $u = 8.6$ ——— ; (b) Phase difference for initial condition $u = 8.6$.

In Figure 4 the approximate amplitude $A(t)$ and phase $y(t) = \theta(t) - \omega_1 t$ (where ω_1 is the lower natural frequency of the linear chain) are depicted for a system with initial conditions zero except $\dot{u}_1(0) = u$. Of interest is to examine the transient evolution of the phase difference $y(t)$ for the case when resonance capture (and energy pumping) occurs ($u = 8.6$, Figure 4b). We divide the evolution of $y(t)$ into three phases. In the first phase when resonance capture occurs there is a small-amplitude fast oscillation of $y(t)$ about a slowly-varying mean, indicating that when

resonance capture occurs in the neighborhood of a natural frequency, there occurs a slow variation of the phase of the nonlinear attachment in the neighborhood of the linearized natural frequency of the resonant mode. In the second phase a rapid transition of $y(t)$ occurs away from the resonant natural frequency, and in third phase $y(t)$ is observed to perform small-amplitude oscillations about a small mean value. A perturbation analysis carried out [8] proves that in the third phase of the response the nonlinear attachment oscillates with a frequency equal to $\sqrt{\varepsilon(1-\varepsilon\beta^2/4)}$ and effective damping ratio equal to $\zeta = \beta\varepsilon/2$. This result is consistent with the numerical instantaneous frequency simulation depicted in Figure 2.

4. A New way for Multi-Frequency Nonlinear Energy Transfer [9,10]

Our next research aim is to study if by using MDOF nonlinear attachments it is possible to extract simultaneously energy from multiple linear modes, through simultaneous dynamic (resonance) interactions of multiple nonlinear normal modes (NNMs) of the attachments with multiple modes the linear system.

In an attempt to answer this question we modify the attachment-linear system configuration, by considering the system of figure 4, composed of a two-DOF linear system, weakly connected to a three-DOF essentially nonlinear attachment.

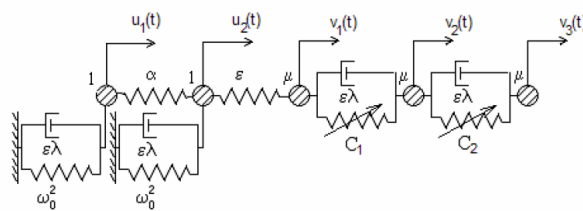
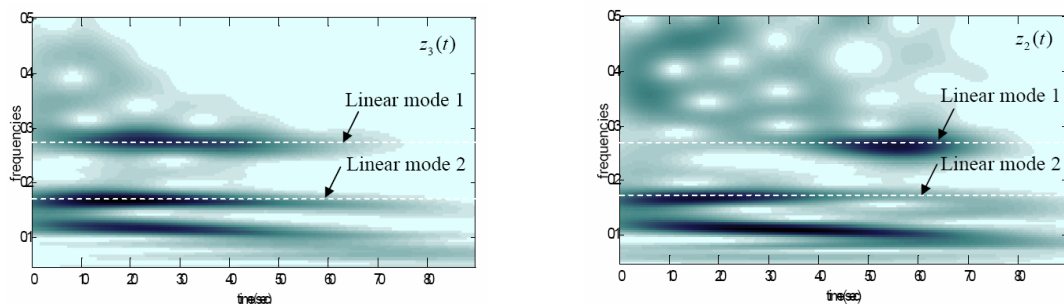


Figure 4. The multi-DOF system under Consideration

The system under consideration is governed by the following system of equations:

$$\begin{aligned}
 \ddot{x}_1 + \varepsilon\lambda\dot{x}_1 + (\omega_1^2 + \frac{\varepsilon}{2})x_1 - \varepsilon(\frac{x_2}{2} + v_1) &= 0 \\
 \ddot{x}_2 + \varepsilon\lambda\dot{x}_2 + (\omega_2^2 + \frac{\varepsilon}{2})x_2 - \varepsilon(\frac{x_1}{2} - v_1) &= 0 \\
 \mu\ddot{v}_1 + \varepsilon\lambda(\dot{v}_1 - \dot{v}_2) + \varepsilon(v_1 + \frac{x_2 - x_1}{2}) + C(v_1 - v_2)^3 &= 0 \\
 \mu\ddot{v}_2 + \varepsilon\lambda(2\dot{v}_2 - \dot{v}_1 - \dot{v}_3) + C(v_2 - v_1)^3 + C(v_2 - v_3)^3 &= 0 \\
 \mu\ddot{v}_3 + \varepsilon\lambda(\dot{v}_3 - \dot{v}_2) + C(v_3 - v_2)^3 &= 0
 \end{aligned} \tag{9}$$

The multi frequency nature of the energy transfer to the nonlinear attachments becomes apparent by studying the frequency-time plots depicted in Figure 5.



(a)

(b)

Figure 5. Wavelet analysis of the transient responses of the nonlinear subsystems of the coupled system (frequencies in Hz). (a) $Z_3(t)$ ____, (b) $Z_2(t)$ ____

These results indicate that MDOF essentially nonlinear attachments can extract energy from linear systems in multi-frequency fashion, through simultaneous dynamic interactions of multiple modes of the nonlinear attachments with multiple modes of the linear system. This form of multi-frequency energy exchange is different than the resonance capture cascades encountered in previous works, where energy extractions to SDOF nonlinear attachments occurs in a sequential manner.

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